

LOCAL AND GLOBAL WELL-POSEDNESS FOR THE CHERN-SIMONS-DIRAC SYSTEM IN ONE DIMENSION

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ABSTRACT. We consider the Cauchy problem for the Chern-Simons-Dirac system on \mathbb{R}^{1+1} with initial data in H^s . Almost optimal local well-posedness is obtained. Moreover, we show that the solution is global in time, provided that initial data for the spinor component has finite charge, or L^2 norm.

1. INTRODUCTION

The Chern-Simons action was first studied from a geometric point of view in [6]. Subsequently, it was proposed as an alternative gauge field theory to the standard Maxwell theory of electrodynamics on Minkowski space \mathbb{R}^{1+2} [8]. As well as being of interest theoretically, it has also been successfully applied to explain phenomena in the physics of planar condensed matter, such as the fractional quantum Hall effect [13]. Recently, much progress has been made on the Cauchy problem for the Chern-Simons action coupled with various other field theories such as Chern-Simons-Higgs, [4, 10], and Chern-Simons-Dirac [10].

In the current article we consider the Cauchy problem for the Chern-Simons-Dirac (CSD) system in \mathbb{R}^{1+1} . This system was first studied by Huh in [11] as a simplified version of the more standard CSD system on \mathbb{R}^{1+2} . The CSD system on \mathbb{R}^{1+1} is given by

$$\begin{aligned} i\gamma^\mu D_\mu \psi &= m\psi \\ \partial_t A_1 - \partial_x A_0 &= \psi^\dagger \alpha \psi \\ \partial_t A_0 - \partial_x A_1 &= 0 \end{aligned} \tag{CSD}$$

with initial data $\psi(0) = f$, $A(0) = a$, where the spinor ψ is a \mathbb{C}^2 valued function of $(t, x) = (x_0, x_1) \in \mathbb{R}^{1+1}$ and the gauge components A_0 and A_1 of the gauge $A = (A_0, A_1)$ are real valued. The covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$ and we raise and lower indices with respect to the metric $g = \text{diag}(1, -1)$. Repeated indices are summed over $\mu = 0, 1$, and we use ψ^\dagger to denote the conjugate transpose of ψ . We

take the standard representation of the Gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and let $\alpha = \gamma^0$.

The system (CSD) is interesting from a mathematical point of view for a number of reasons. Firstly solutions to (CSD) satisfy conservation of charge, i.e. we have $\|\psi(t)\|_{L^2} = \|f\|_{L^2}$ for any $t \in \mathbb{R}$. This is similar to the Dirac-Klein-Gordon (DKG) equation where conservation of charge also holds. We remark that conservation of charge forms a crucial component in the study of global existence for DKG [17, 19]. On the other hand, conservation of charge fails for other quadratic Dirac equations which have been studied in the literature [3, 15, 16]. Secondly, there is substantial null structure in the nonlinear terms in (CSD), in the sense that (CSD) is roughly equivalent to a system of nonlinear wave equations of the form

$$\square \Psi = Q(\Psi, \Psi)$$

where $Q(\Psi, \Psi)$ is a combination of the null forms $Q_{ij} = \partial_i \Psi_\mu \partial_j \Psi_\nu - \partial_j \Psi_\mu \partial_i \Psi_\nu$ and $Q_0 = g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi$. Moreover the structure of the equation means that in the mass free case $m = 0$, the spinor ψ can be explicitly solved in terms of the initial data ψ_0 and the gauge A . This idea was used in [11] to derive a number of interesting observations on the asymptotic behaviour of solutions to (CSD) as $t \rightarrow \infty$.

Currently the best known results for the Cauchy problem for (CSD) are due to Huh in [11] where it was shown that the (CSD) system is locally well-posed for initial data in the charge class $(\psi_0, a_0) \in L^2 \times L^2$, and globally well-posed for $(\psi_0, a_0) \in H^1 \times H^1$. To prove the local in time result, Huh rewrote (CSD) as a system of nonlinear wave equations and showed that the nonlinear terms contained null structure. The null form estimates of Klainerman and Machedon [12] then completed the proof.

In the current article we use a different approach. Instead of rewriting (CSD) as a wave equation, we factor the Dirac and Gauge components into null-coordinates $x \pm t$ and use Sobolev spaces adapted to these coordinates. In one space dimension, Sobolev spaces based on null coordinates seem to behave better than the closely related $X_{\pm}^{s,b}$ type spaces of Bourgain-Klainerman-Machedon which have been used in many other low-regularity results on Dirac equations in one dimension, see for instance the results in [5, 14]. Our main result is the following.

Theorem 1. *Let $\frac{-1}{2} < r \leq s \leq r + 1$ and $(f, a) \in H^s \times H^r$. Then there exists $T > 0$ and a solution $(\psi, A) \in C([-T, T], H^s \times H^r)$ to (CSD). Moreover solution depends continuously on the initial data, is unique in some subspace of $C([-T, T], H^s \times H^r)$, and any additional regularity persists in time¹.*

Remark 1. If we set $m = 0$, then solutions to (CSD) are invariant under the scaling $(u, A) \mapsto \frac{1}{\lambda}(u, A)(\frac{t}{\lambda}, \frac{x}{\lambda})$. Hence the scale invariant space is $\dot{H}^{-\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$. Since we do not expect any well-posedness below the scaling regularity, the range of well-posedness in Theorem 1 is essentially optimal, except possibly at the endpoint $r = \frac{-1}{2}$. Moreover, it should be possible to show that (CSD) is ill-posed in some sense outside of the range given in Theorem 1 by using the techniques in [14], but we do not consider the problem of ill-posedness here.

¹ More precisely, if $(\psi_0, a_0) \in H^{s'} \times H^{r'}$ with $s' \geq s$, $r' \geq r$, and $r' \leq s' \leq r' + 1$, then we can conclude that $(\psi, A) \in C([-T, T], H^{s'} \times H^{r'})$, where T only depends on the size of $\|f\|_{H^s} + \|a\|_{H^r}$.

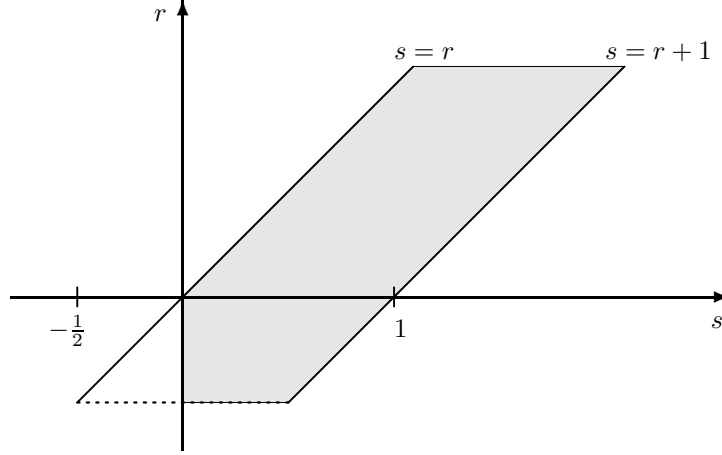


FIGURE 1. The domain of local/global well-posedness from Theorem 1 and Corollary 2. We have local existence inside the lines $s = r$ and $s = r + 1$ for $r > -\frac{1}{2}$. Global existence holds inside the shaded region.

The local existence portion of Theorem 1 will follow by the standard iteration argument, using estimates contained in [14]. The proof of uniqueness is more difficult and does not follow directly from the existence proof, primarily because the spaces used to prove existence do not scale nicely on the domain $[-T, T] \times \mathbb{R}$. Instead we will need to prove a more precise version of an energy inequality from [14]. See Proposition 10 below. Finally the persistence of regularity is quite interesting as it allows both the regularity of the spinor, ψ , and the gauge, A , to be varied independently, provided that we remain in the region of well-posedness.

We now turn to the question of global well-posedness. In the case $s \geq 0$ we can exploit the conservation of charge together with a decomposition argument from [5] to obtain the following.

Corollary 2. *Assume that $s \geq 0$ in Theorem 1. Then the local solution can be extended to a global solution $(\psi, A) \in C(\mathbb{R}, H^s \times H^r)$.*

We now give a brief outline of this article. In Section 2 we gather together the estimates we require in the proof of Theorem 1. The local existence component of Theorem 1 is proven in Section 3. The proof of uniqueness is contained in Section 4. Finally in Section 5 we prove Corollary 2.

Notation. Throughout this paper C denotes a positive constant which can vary from line to line. The notation $a \lesssim b$ denotes the inequality $a \leq Cb$. We let $L^p(\mathbb{R}^n)$ denote the usual Lebesgue space. Occasionally we write $L^p(\mathbb{R}^n) = L^p$ when we can do so without causing confusion. This comment also applies to the other function spaces which appear throughout this paper. If X is a metric space and $I \subset \mathbb{R}$ is an interval, then $C(I, X)$ denotes the set of continuous functions from I into X . For $s \in \mathbb{R}$, we define H^s to be the usual Sobolev space defined using the norm

$$\|f\|_{H^s(\mathbb{R})} = \|\langle \xi \rangle^s \hat{f}\|_{L^2(\mathbb{R})}$$

where \hat{f} denotes the Fourier transform of f and $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. The space-time Fourier transform of a function $\psi(t, x)$ is denoted by $\tilde{\psi}(\tau, \xi)$. We also use the notation $\mathcal{F}_y(f)$ to denote the Fourier transform of f with respect to the variable y .

If X is a Banach space of functions defined on \mathbb{R}^n , then for an open set $\Omega \subset \mathbb{R}^n$ we define the restriction space $X(\Omega)$ by restricting elements of X to Ω . If we equip $X(\Omega)$ with the norm

$$\|f\|_{X(\Omega)} = \inf_{g=f \text{ on } \Omega} \|g\|_X$$

then $X(\Omega)$ is also a Banach space. Finally, for $a, b, c \in \mathbb{R}$ we use the notation $c \prec \{a, b\}$ to denote that either

$$a + b \geq 0, \quad c \leq \min\{a, b\}, \quad c < a + b - \frac{1}{2}$$

or

$$a + b > 0, \quad c < \min\{a, b\}, \quad c \leq a + b - \frac{1}{2}$$

holds. Note that $c \prec \{a, b\}$ implies that the following product inequality for Sobolev spaces holds

$$\|fg\|_{H^c(\mathbb{R})} \lesssim \|f\|_{H^a(\mathbb{R})} \|g\|_{H^b(\mathbb{R})}.$$

2. ESTIMATES

The main estimates we require in the proof of Theorem 1 have already been proven in [14]. Define

$$\|u\|_{Z_{\pm}^{s,b}} = \|\langle \tau \mp \xi \rangle^s \langle \tau \pm \xi \rangle^b \tilde{\psi}(\tau, \xi)\|_{L_{\tau, \xi}^2}.$$

Note that $Z_{\pm}^{s,b}$ is just the product Sobolev space in the null directions $x \pm t$. The $Z_{\pm}^{s,b}$ space is enough to control the nonlinear terms in (CSD). However for s close to $\frac{-1}{2}$, the space $Z_{\pm}^{s,b}$ is not contained inside $C(\mathbb{R}, H^s(\mathbb{R}))$. Thus to prove the local well-posedness result in Theorem 1, we will need to add a component to control the $L_t^\infty H^s$ norm. To this end, following [14], we define the space $Y_{\pm}^{s,b}$ by using the norm

$$\|u\|_{Y_{\pm}^{s,b}} = \|\langle \xi \rangle^s \langle \tau \pm \xi \rangle^b \tilde{u}(\tau, \xi)\|_{L_\xi^2 L_\tau^1}.$$

It is easy to see that

$$\|u\|_{L_t^\infty H_x^s} \leq \|u\|_{Y_{\pm}^{s,0}}.$$

and so $Z_{\pm}^{s,b} \cap Y_{\pm}^{s,0} \subset C(\mathbb{R}, H^s(\mathbb{R}))$. We remark that spaces of the form $Y_{\pm}^{s,b}$ have been used previously to augment the standard $X^{s,b}$ spaces for $b = \frac{1}{2}$ in the periodic case in [2], see also [9].

The first result we will need is the following energy type inequality.

Lemma 3 ([14] Lemma 3.2). *Let $s, b \in \mathbb{R}$ and $S = [-1, 1] \times \mathbb{R}$. Suppose u is a solution to*

$$\begin{aligned} \partial_t u \pm \partial_x u &= F \\ u(0) &= f \end{aligned}$$

on S . Then

$$\|u\|_{Z_{\pm}^{s,b}(S)} + \|u\|_{Y_{\pm}^{s,0}(S)} \lesssim \|f\|_{H^s} + \inf_{F'|_S = F} \left(\|F'\|_{Z_{\pm}^{s,b-1}} + \|F'\|_{Y_{\pm}^{s,-1}} \right) \quad (1)$$

where the infimum is over all $F' \in Z_{\pm}^{s,b-1} \cap Y_{\pm}^{s,-1}$ with $F' = F$ on S .

The previous energy inequality is sufficient to prove existence of solutions to (CSD), however to obtain uniqueness we will require a slightly more refined version of Lemma 3 which we leave to Section 4.

To close the iteration argument we will need the following nonlinear estimate contained in [14].

Lemma 4 ([14], Lemma 3.4). *Let $s_1, s_2, b_1, b_2, s \in \mathbb{R}$ and assume there exists $a_0, b_0 \in \mathbb{R}$ such that*

$$\begin{aligned} a_0 < \{s_1, b_2\}, \quad b_0 < \{s_2, b_1\}, \quad s < \{a_0, b_0 + 1\} \\ s_1 + b_1 > \frac{-1}{2}, \quad s_2 + b_2 > \frac{-1}{2}. \end{aligned} \quad (2)$$

Then we have

$$\|uv\|_{Y_{\pm}^{s,-1}} \lesssim \|u\|_{Z_{\pm}^{s_1, b_1}} \|v\|_{Z_{\mp}^{s_2, b_2}}.$$

We also have the following well known product estimates for Sobolev spaces.

Lemma 5. *Assume $s < \{s_1, b_2\}$ and $b < \{b_1, s_2\}$. Then*

$$\|uv\|_{Z_{\pm}^{s, b}} \lesssim \|u\|_{Z_{\pm}^{s_1, b_1}} \|v\|_{Z_{\mp}^{s_2, b_2}}.$$

Finally we will need the following Lemma which will help simplify the arguments leading to uniqueness.

Lemma 6. *Let $\frac{-1}{2} < s < \frac{1}{2}$ and $0 < T < 1$. Assume $\rho \in H^1$ and let $\rho_T(t) = \rho(\frac{t}{T})$. Then*

$$\|\rho_T(t)f(t)\|_{H_t^s} \lesssim_{\rho} \|f\|_{H^s} \quad (3)$$

with constant independent of T . Consequently

$$\|\rho_T(t)u\|_{Z_{\pm}^{s, 0}} \lesssim_{\rho} \|u\|_{Z_{\pm}^{s, 0}} \quad (4)$$

with constant independent of T .

Proof. The inequality (3) is well-known. For the convenience of the reader we sketch the proof in the appendix. To prove (4) we use a change of variables

$$\begin{aligned} \|\rho_T(t)\psi\|_{Z_{\pm}^{s, 0}} &= \left\| \langle \tau \mp \xi \rangle^s \int \widehat{\rho_T}(\lambda) \widetilde{\psi}(\tau - \lambda, \xi) d\lambda \right\|_{L_{\tau, \xi}^2} \\ &= \left\| \langle \tau \rangle^s \int \widehat{\rho_T}(\lambda) \widetilde{\psi}(\tau \pm \xi - \lambda, \xi) d\lambda \right\|_{L_{\tau, \xi}^2} \end{aligned}$$

and then apply (3). \square

3. LOCAL EXISTENCE

We start by noting that if we let $u_{\pm} = \psi_1 \pm \psi_2$ and $A_{\pm} = A_0 \mp A_1$, we can rewrite (CSD) in the form

$$\begin{aligned} i(\partial_t u_+ + \partial_x u_+) &= m u_- - A_- u_+ \\ i(\partial_t u_- - \partial_x u_-) &= m u_+ - A_+ u_- \\ u_{\pm}(0) &= f_{\pm} \end{aligned} \quad (5)$$

and

$$\begin{aligned} \partial_t A_+ + \partial_x A_+ &= -\Re(u_+ \overline{u_-}) \\ \partial_t A_- - \partial_x A_- &= \Re(u_+ \overline{u_-}) \\ A_{\pm}(0) &= a_{\pm} \end{aligned} \quad (6)$$

where $f_{\pm} = f_1 \pm f_2$, $a_{\pm} = a_0 \mp a_1$, and we use $\Re(z)$ to denote the real part of $z \in \mathbb{C}$. The formulation (5), (6) is much easier to work with than (CSD) as the null structure is more apparent. Namely all the nonlinear terms involve products of the form $\psi_+ \phi_-$ which behave far better than the product $\psi_+ \phi_+$, see for instance the estimates in [18]. The fact that the nonlinear terms in (5) and (6) are all $+$ -products

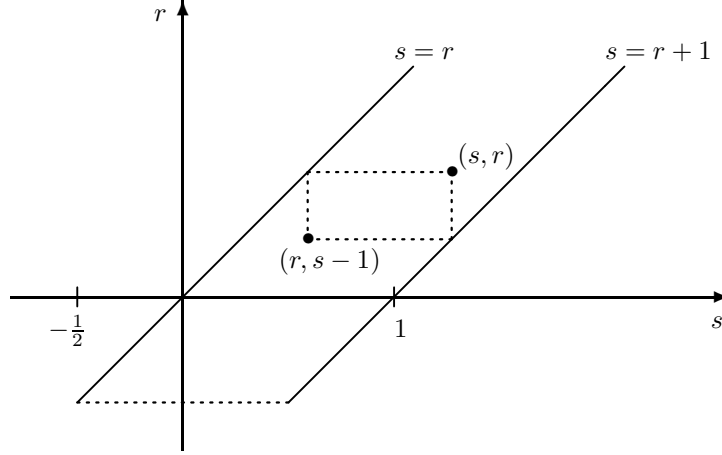


FIGURE 2. The time of existence given by the rescaled version of Theorem 7 at regularity $H^s \times H^r$, only depends on the size of the initial data at the regularity $H^r \times H^{s-1}$ (provided $s - 1 > \frac{-1}{2}$).

is a reflection of the null structure present in the (CSD) system.

We will deduce Theorem 1 from the following.

Theorem 7. *Let $\frac{-1}{2} < r \leq s \leq r + 1$ and assume $f \in H^s$, $a \in H^r$. Choose $r^* > \frac{-1}{2}$ with $s - 1 \leq r^* \leq r$. Then there exists $\epsilon > 0$ such that if $|m| < \epsilon$ and*

$$\|f\|_{H^r} + \|a\|_{H^{r^*}} < \epsilon$$

then there exists a solution $(\psi, A) \in C([-1, 1], H^s \times H^r)$ to (CSD) with $(\psi, A)(0) = (f, a)$. Moreover solution depends continuously on the initial data and if we let $u_{\pm} = \psi_1 \pm \psi_2$ and $A_{\pm} = A_0 \mp A_1$ then

$$u_{\pm} \in Z_{\pm}^{s,b}(S) \cap Y_{\pm}^{s,-1}(S), \quad A_{\pm} \in Z_{\pm}^{r,b}(S) \cap Y_{\pm}^{s,-1}(S)$$

for any $b > \frac{1}{2}$ with $s \leq b \leq r^ + 1$ and $S = [-1, 1] \times \mathbb{R}$.*

Assume for the moment that Theorem 7 holds, we deduce Theorem 1 as follows. Let $(f, a) \in H^s \times H^r$ with $\frac{-1}{2} < r \leq s \leq r + 1$. Theorem 7 together with a scaling argument then gives a solution $(\psi, A) \in C([-T, T], H^s \times H^r)$ that depends continuously on the initial data, where T only depends on some negative power of $\|f\|_{H^r} + \|a\|_{H^{r^*}}$ with $r^* > \frac{-1}{2}$ and $s - 1 \leq r^* \leq r$, see Figure 2. The uniqueness we leave till the next section. Hence to complete the proof of Theorem 1 it only remains to check that any additional regularity persists in time.

Suppose the initial data has additional smoothness $(f, a) \in H^{s^*} \times H^{r^*}$ with $s^* > s$, $r^* > r$, and $r^* \leq s^* \leq r^* + 1$. Applying the local existence result we have $(\psi, A) \in C((-T^*, T^*), H^{s^*} \times H^{r^*})$ for some $T^* > 0$. Persistence of regularity will follow if we can obtain $T^* \geq T$. To this end, we note that it is enough to show that if $T^* < T$ and

$$\limsup_{t \rightarrow T^*} (\|\psi(t)\|_{H^{s^*}} + \|A(t)\|_{H^{r^*}}) = \infty \quad (7)$$

then we also have

$$\limsup_{t \rightarrow T^*} (\|\psi(t)\|_{H^s} + \|A(t)\|_{H^r}) = \infty. \quad (8)$$

This is done in steps as follows. We first deduce by the rescaled version of Theorem 7 that

$$\limsup_{t \rightarrow T^*} (\|\psi(t)\|_{H^{r^*}} + \|A(t)\|_{H^{\max\{s^*-1, \frac{-1}{2}+\epsilon\}}}) = \infty \quad (9)$$

for any sufficiently small $\epsilon > 0$. Since if not, then we can choose some sequence of points $t_n \rightarrow T^*$ with $\sup_n \|\psi(t_n)\|_{H^{r^*}} + \|A(t_n)\|_{H^{\max\{s^*-1, \frac{-1}{2}+\epsilon\}}} < \infty$. Taking t_n sufficiently close to T and applying a rescaled version of Theorem 7 with initial data $(\psi(t_n), A(t_n))$, we can extend our solution beyond T^* , contradicting (7). Thus provided $T^* < \infty$ and (7) holds, we must have (9).

Repeating this argument again with (7) replaced with (9) we obtain

$$\limsup_{t \rightarrow T^*} (\|\psi(t)\|_{H^{\max\{s^*-1, \frac{-1}{2}+\epsilon\}}} + \|A(t)\|_{H^{\max\{r^*-1, \frac{-1}{2}+\epsilon\}}}) = \infty.$$

We now continue in this manner and observe that after k iterations, the $H^{\max\{s^*-k, \frac{-1}{2}+\epsilon\}} \times H^{\max\{r^*-k, \frac{-1}{2}+\epsilon\}}$ norm must blowup as we approach T^* . Taking k such that $s^* - k \leq s$ and $r^* - k \leq r$ we obtain (8) as required.

We now come to the proof of small data local well-posedness for (CSD).

Proof of Theorem 7. Let $\frac{-1}{2} < r \leq s \leq r+1$ and choose $b > \frac{1}{2}$ with $s \leq b \leq r^*+1$. Note that this is possible since $r^* \geq s-1$ and $r^* > \frac{-1}{2}$. Let $r \leq s' \leq s$. We claim that Lemma 4 and Lemma 5 imply the estimates

$$\|uv\|_{Y_{\pm}^{r,-1}} \leq \|uv\|_{Y_{\pm}^{s',-1}} \lesssim \|u\|_{Z_{\pm}^{s',b}} \|v\|_{Z_{\mp}^{r^*,b}} \quad (10)$$

and

$$\|uv\|_{Z_{\pm}^{r,b-1}} \leq \|uv\|_{Z_{\pm}^{s',b-1}} \lesssim \|u\|_{Z_{\pm}^{s',b}} \|v\|_{Z_{\mp}^{r^*,b}}. \quad (11)$$

To obtain the estimate (10), an application of Lemma 4 reduces the problem to showing that there exists $a_0, b_0 \in \mathbb{R}$ such that

$$\begin{aligned} a_0 < \{s', b\}, \quad b_0 < \{r^*, b\}, \quad s' < \{a_0, b_0 + 1\} \\ s' + b > \frac{-1}{2}, \quad r^* + b > \frac{-1}{2}. \end{aligned}$$

Since $r^* \leq r \leq s' \leq s \leq b$, we let $a_0 = s'$, $b_0 = r^*$. It is clear that $s' < \{s', b\}$ and $r^* < \{r^*, b\}$. Thus the only remaining conditions are

$$s' + r^* + 1 \geq 0, \quad s' \leq r^* + 1, \quad s' < s' + r^* + 1 - \frac{1}{2}.$$

But these also hold provided $r^*, s' > \frac{-1}{2}$ and $s' \leq r^* + 1$, which follows since $s' \leq s \leq r^* + 1$. Consequently (10) holds.

The remaining estimate, (11), follows from Lemma 5 provided that

$$s' < \{s', b\}, \quad b-1 < \{r^*, b\}.$$

Using the assumptions $s', r^* > \frac{-1}{2}$ and $b > \frac{1}{2}$ this reduces to

$$\begin{aligned} s' &\leq b, & s' &< s' + b - \frac{1}{2} \\ b-1 &\leq r^*, & b-1 &< r^* + b - \frac{1}{2}. \end{aligned}$$

These inequalities also hold in view of the assumptions $\frac{-1}{2} < s' \leq b$ and $\frac{1}{2} < b \leq r^* + 1$. Therefore (10) and (11) both hold.

It suffices to consider the system (5) and (6) with the assumption

$$\sum_{\pm} \|f_{\pm}\|_{H^r} + \|a_{\pm}\|_{H^{r*}} < \epsilon.$$

Let $S = [-1, 1] \times \mathbb{R}$ and define the Banach space $E^s = \{v = (v_+, v_-) \mid v_{\pm} \in Z_{\pm}^{s,b}(S) \cap Y_{\pm}^{s,0}(S)\}$ with norm

$$\|v\|_{E^s} = \sum_{\pm} \|v_{\pm}\|_{Y_{\pm}^{s,0}(S)} + \|v_{\pm}\|_{Z_{\pm}^{s,b}(S)}$$

Note that since $Y_{\pm}^{s,0}(S) \subset L_t^{\infty} H_x^s(S)$ we have $\|v\|_{L_t^{\infty} H_x^s(S)} \lesssim \|v\|_{E^s}$. Let $\Gamma = \sum_{\pm} \|f_{\pm}\|_{H^s} + \|a_{\pm}\|_{H^r}$ and define the closed subset $\mathcal{X}_{\epsilon} \subset E^s \times E^r$ by

$$\mathcal{X}_{\epsilon} = \{\|u\|_{E^r} + \|A\|_{E^{r*}} \leq 2C\epsilon\} \cap \{\|u\|_{E^s} + \|A\|_{E^r} \leq 2C\Gamma\}.$$

Define the map $\mathcal{S} : \mathcal{X}_{\epsilon} \rightarrow \mathcal{X}_{\epsilon}$ by letting $\mathcal{S}(u, A) = (v, B)$ be the solution to

$$\begin{aligned} i(\partial_t \pm \partial_x)v_{\pm} &= mu_{\mp} + A_{\mp}u_{\pm} \\ (\partial_t \pm \partial_x)B_{\pm} &= \pm \Re(u_+ \bar{u}_-) \\ v_{\pm}(0) &= f_{\pm}, \quad B_{\pm}(0) = a_{\pm}. \end{aligned} \tag{12}$$

Then using Lemma 3 together with (10) and (11) we obtain

$$\|v\|_{E^s} + \|B\|_{E^r} \lesssim \sum_{\pm} (\|f_{\pm}\|_{H^s} + \|a_{\pm}\|_{H^r}) + |m|(\|u\|_{E^s} + \|A\|_{E^r}) + (\|u\|_{E^r} + \|A\|_{E^{r*}})(\|u\|_{E^s} + \|A\|_{E^r})$$

and

$$\|v\|_{E^r} + \|B\|_{E^{r*}} \lesssim \sum_{\pm} (\|f_{\pm}\|_{H^r} + \|a_{\pm}\|_{H^{r*}}) + |m|(\|u\|_{E^r} + \|A\|_{E^{r*}}) + (\|u\|_{E^r} + \|A\|_{E^{r*}})^2.$$

The assumption $(u, A) \in \mathcal{X}_{\epsilon}$ then gives the inequalities

$$\begin{aligned} \|v\|_{E^s} + \|B\|_{E^r} &\leq C\Gamma + C\epsilon\Gamma + C^2\epsilon\Gamma \\ \|v\|_{E^r} + \|B\|_{E^{r*}} &\leq C\epsilon + C\epsilon^2 + C^2\epsilon^2 \end{aligned}$$

Therefore, provided ϵ is sufficiently small, depending only on the constants in (10), (11), and (1), we see that \mathcal{S} is well defined. A similar argument shows that \mathcal{S} is a contraction mapping, consequently we have existence, uniqueness in \mathcal{X}_{ϵ} , and continuous dependence on the initial data. \square

4. UNIQUENESS

In this section we will complete the proof of Theorem 1 and show that the solution obtained in Section 3 is unique. More precisely, we will prove the following.

Proposition 8. *Let $\frac{-1}{2} < r \leq s \leq r+1$, $T > 0$, and $b > \frac{1}{2}$. Define $S_T = [-T, T] \times \mathbb{R}$. Assume (u, A) and (v, B) are solutions to (5) and (6) with $u_{\pm}, v_{\pm} \in Z_{\pm}^{s,b}(S_T)$ and $A_{\pm}, B_{\pm} \in Z_{\pm}^{r,b}(S_T)$. If $(u, A)(0) = (v, B)(0)$ then $(u, A) = (v, B)$ on S_T .*

The proof of Proposition 8 is slightly involved as we need to understand the behaviour of the energy inequality Lemma 3 on the domain S_T for small T . For the $Y^{s,b}$ component this is reasonably straightforward.

Lemma 9. *Let $s \in \mathbb{R}$, $0 < T < 1$, and $0 < \epsilon < 1$. Suppose ψ is a solution to*

$$\begin{aligned}\partial_t \psi \pm \partial_x \psi &= F \\ \psi(0) &= f.\end{aligned}$$

Let $\rho \in C_0^\infty$ and define $\rho_T(t) = \rho(\frac{t}{T})$. Then

$$\|\rho_T(t)\psi\|_{Y_{\pm}^{s,0}} \lesssim_\rho \|f\|_{H^s} + \|\langle \xi \rangle^s \min\{T, |\tau \pm \xi|^{-1}\} \tilde{F}\|_{L_\xi^2 L_\tau^1} \quad (13)$$

$$\lesssim \|f\|_{H^s} + T^\epsilon \|F\|_{Y_{\pm}^{s,\epsilon-1}} \quad (14)$$

with constant independent of T .

Proof. It is easy to see that (13) follows from the estimate

$$\left\| \mathcal{F}_t \left[\rho_T(t) \int_0^t e^{\mp i(t-s)\xi} \widehat{F}(s) ds \right] (\tau, \xi) \right\|_{L_\xi^2 L_\tau^1} \lesssim \left\| \min\{T, |\tau \pm \xi|^{-1}\} \tilde{F} \right\|_{L_\xi^2 L_\tau^1}. \quad (15)$$

Note that by scaling it is sufficient to consider the case $T = 1$. Consequently $\min\{1, |\tau \pm \xi|^{-1}\} \approx \langle \tau \pm \xi \rangle^{-1}$ and so (15) follows from Lemma 3.2 in [14]. The remaining inequality (14) then follows by observing that since $0 < T < 1$,

$$\min\{T, |\tau \pm \xi|^{-1}\} \lesssim T^\epsilon \langle \tau \pm \xi \rangle^{\epsilon-1}.$$

□

It remains to control $Z_{\pm}^{s,b}$ component of the energy inequality. This is significantly more difficult as both multipliers $\langle \tau + \xi \rangle$ and $\langle \tau - \xi \rangle$ involve the time variable. This observation, together with the fact that $Y^{s,0}$ has a different scaling to $Z^{s,b}$, is the main difficulty in the following proposition.

Proposition 10. *Let $\frac{-1}{2} < s \leq 0$ and $0 < T < 1$. Choose $0 < \epsilon < \frac{1}{2}$ and let $\frac{1}{2} < b < \min\{1 + s, 1 - \epsilon\}$. Assume $\rho, \sigma \in C_0^\infty$ with $\rho(t) = 1$ for $t \in [-1, 1]$, $\sigma(t) = 1$ for $t \in \text{supp } \rho$, and*

$$\text{supp } \rho \subset \text{supp } \sigma \subset [-2, 2].$$

Define $\rho_T(t) = \rho(\frac{t}{T})$ and $\sigma_T(t) = \sigma(\frac{t}{T})$. Let ψ be a solution to

$$\partial_t \psi \pm \partial_x \psi = F.$$

Then

$$\|\rho_T(t)\psi\|_{Z_{\pm}^{s,b}} \lesssim T^{\frac{1}{2}-b} \|\sigma_T(t)\psi\|_{Y_{\pm}^{s,0}} + T^\epsilon \|F\|_{Z_{\pm}^{s,b-1+\epsilon}} \quad (16)$$

with the implied constant independent of T .

Proof. We only prove the $+$ case as the $-$ case is similar. Note that since $\sigma_T(t) = 1$ on $\text{supp } \rho_T$ we may simply write $\psi = \sigma_T \psi$. Let $\Omega \subset \mathbb{R}^2$ and define

$$I(\Omega) = \left\| \langle \tau + \xi \rangle^b \langle \tau - \xi \rangle^s \int_{\mathbb{R}} \widehat{\rho_T}(\tau - \lambda) \tilde{\psi}(\lambda, \xi) d\lambda \right\|_{L_{\tau,\xi}^2(\Omega)}.$$

We break \mathbb{R}^2 into different regions and estimate each region separately. We first consider the set

$$\Omega_1 = \{|\tau + \xi| \leq T^{-1}\}$$

and split this into the regions $2|\tau - \xi| \geq |\xi|$ and $2|\tau - \xi| \leq |\xi|$. In the former region, since $s \leq 0$ and $\langle \tau + \xi \rangle^b \leq T^{-b}$,

$$\begin{aligned} I(\Omega_1 \cap \{2|\tau - \xi| \geq |\xi|\}) &\lesssim T^{-b} \left\| \int_{\mathbb{R}} \widehat{\rho_T}(\tau - \lambda) \langle \xi \rangle^s \widetilde{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}} \\ &\lesssim T^{-b} \|\widehat{\rho_T}(\tau)\|_{L^2_{\tau}} \|\psi\|_{Y_+^{s,0}} \\ &\lesssim_{\rho} T^{\frac{1}{2}-b} \|\psi\|_{Y_+^{s,0}}. \end{aligned}$$

On the other hand if $2|\tau - \xi| \leq |\xi|$ then $|\tau| \approx |\xi| \approx |\tau + \xi| \lesssim T^{-1}$. Hence

$$\begin{aligned} I(\Omega_1 \cap \{2|\tau - \xi| \leq |\xi|\}) &\lesssim \left\| \langle \tau + \xi \rangle^{b-s} \langle \tau - \xi \rangle^s \int \widehat{\rho_T}(\tau - \lambda) \langle \xi \rangle^s \widehat{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(|\tau - \xi|, |\tau + \xi| \lesssim T^{-1})} \\ &\lesssim T^{s-b} \|\widehat{\rho_T}\|_{L^\infty_{\rho}} \|\langle \tau \rangle^s\|_{L^2_{\tau}(|\tau| \lesssim T^{-1})} \|\psi\|_{Y_+^{s,0}} \\ &\lesssim_{\rho} T^{s-b} \times T \times T^{-\frac{1}{2}-s} \|\psi\|_{Y_+^{s,0}} \\ &= T^{\frac{1}{2}-b} \|\psi\|_{Y_+^{s,0}}. \end{aligned}$$

Therefore

$$I(\Omega_1) \lesssim T^{\frac{1}{2}-b} \|\psi\|_{Y_+^{s,0}}.$$

We now consider the region $\Omega_2 = \{|\tau + \xi| \geq T^{-1}\}$. Note that

$$\begin{aligned} (\rho_T(t)\psi)^{\sim}(\tau, \xi) &= \frac{1}{i(\tau + \xi)} \int i((\tau - \lambda) + (\lambda + \xi)) \widehat{\rho_T}(\tau - \lambda) \widetilde{\psi}(\lambda, \xi) d\lambda \\ &= \frac{1}{i(\tau + \xi)} \left[T^{-1} ((\partial_t \rho)_T \psi)^{\sim}(\tau, \xi) + (\rho_T F)^{\sim}(\tau, \xi) \right] \end{aligned}$$

and so, using the fact that $|\tau + \xi| \geq T^{-1} \gg 1$ implies $|\tau + \xi| \approx \langle \tau + \xi \rangle$, we have

$$I(\Omega_2) \leq T^{-1} \left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s ((\partial_t \rho)_T \psi)^{\sim} \right\|_{L^2_{\tau, \xi}(\Omega_2)} + \left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s (\rho_T F)^{\sim} \right\|_{L^2_{\tau, \xi}(\Omega_2)}. \quad (17)$$

We estimate each of these terms separately. For the first term we follow the Ω_1 case and decompose Ω_2 into $2|\tau - \xi| \geq |\xi|$ and $2|\tau - \xi| \leq |\xi|$. In the former region we use the fact that $\langle \tau + \xi \rangle^{b-1} \leq T^{1-b}$ to deduce that

$$\begin{aligned} T^{-1} \left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s \int \widehat{(\partial_t \rho)_T}(\tau - \lambda) \widetilde{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2 \cap \{2|\tau - \xi| \geq |\xi|\})} \\ \lesssim T^{-b} \left\| \int \widehat{(\partial_t \rho)_T}(\tau - \lambda) \langle \xi \rangle^s \widetilde{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}} \\ \lesssim T^{-b} \|\widehat{(\partial_t \rho)_T}\|_{L^2} \|\psi\|_{Y_+^{s,0}} \\ \lesssim_{\rho} T^{\frac{1}{2}-b} \|\psi\|_{Y_+^{s,0}}. \end{aligned}$$

On the other hand for $2|\tau - \xi| \leq |\xi|$ we have $|\tau + \xi| \approx |\xi|$ and so

$$\begin{aligned} T^{-1} \left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s \int \widehat{(\partial_t \rho)_T}(\tau - \lambda) \widetilde{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2 \cap \{2|\tau - \xi| \leq |\xi|\})} \\ \lesssim T^{-1} \left\| \langle \tau + \xi \rangle^{b-1-s} \langle \tau - \xi \rangle^s \int \widehat{(\partial_t \rho)_T}(\tau - \lambda) \langle \xi \rangle^s \widetilde{\psi}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)} \\ \lesssim T^{s-b} \|\psi\|_{Y_+^{s,0}} \sup_{\xi, \lambda} \|\langle \tau - \xi \rangle^s \widehat{(\partial_t \rho)_T}(\tau - \lambda)\|_{L^2_{\tau}}. \end{aligned}$$

To control the $\partial_t \rho$ term we use

$$\begin{aligned} \|\langle \tau - \xi \rangle^s \widehat{(\partial_t \rho)_T}(\tau - \lambda)\|_{L^2_\tau} &\lesssim \|\langle \tau - \xi \rangle^s \widehat{(\partial_t \rho)_T}(\tau - \lambda)\|_{L^2_\tau(|\tau - \xi| \leq T^{-1})} + \|\langle \tau - \xi \rangle^s \widehat{(\partial_t \rho)_T}(\tau - \lambda)\|_{L^2_\tau(|\tau - \xi| \geq T^{-1})} \\ &\lesssim \|\langle \tau \rangle^s\|_{L^2_\tau(|\tau| \leq T^{-1})} \|\widehat{(\partial_t \rho)_T}\|_{L^\infty} + T^{-s} \|\partial_t \rho_T\|_{L^2} \\ &\lesssim_\rho T^{\frac{1}{2}-s} \end{aligned}$$

and so we can estimate the first term in (17).

Finally, to estimate the remaining term in (17), we write

$$\begin{aligned} &\left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s \int \widehat{\rho_T}(\lambda - \tau) \widetilde{F}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)} \\ &\lesssim T^\epsilon \left\| \langle \tau + \xi \rangle^{b-1+\epsilon} \langle \tau - \xi \rangle^s \int_{2|\tau + \xi| \leq |\lambda + \xi|} \widehat{\rho_T}(\lambda - \tau) \widetilde{F}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)} \\ &\quad + T^\epsilon \left\| \langle \tau + \xi \rangle^{b-1+\epsilon} \langle \tau - \xi \rangle^s \int_{2|\tau + \xi| \geq |\lambda + \xi|} \widehat{\rho_T}(\lambda - \tau) \widetilde{F}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)}. \end{aligned}$$

In the region $2|\tau + \xi| \leq |\lambda + \xi|$ we have $|\lambda + \xi| \approx |\tau - \lambda|$ and so, using the fact that $|\tau + \xi| \geq T^{-1}$,

$$\begin{aligned} &\left\| \langle \tau + \xi \rangle^{b-1+\epsilon} \langle \tau - \xi \rangle^s \int_{2|\tau + \xi| \leq |\lambda + \xi|} \widehat{\rho_T}(\lambda - \tau) \widetilde{F}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)} \\ &\lesssim \left\| \langle \tau - \xi \rangle^s \int (T|\tau - \lambda|)^{1-b-\epsilon} |\widehat{\rho_T}(\tau - \lambda)| \langle \lambda + \xi \rangle^{b-1+\epsilon} |\widetilde{F}(\lambda, \xi)| d\lambda \right\|_{L^2_{\tau, \xi}} \\ &\lesssim_\rho \|F\|_{Z_+^{s, b-1+\epsilon}} \end{aligned}$$

where the last line follows from an application of Lemma 6. On the other hand, if $2|\tau + \xi| \geq |\lambda + \xi|$, we can simply use the estimate $\langle \tau + \xi \rangle^{b-1+\epsilon} \lesssim \langle \lambda + \xi \rangle^{b-1+\epsilon}$ followed by another application of Lemma 6. Therefore we have

$$\left\| \langle \tau + \xi \rangle^{b-1} \langle \tau - \xi \rangle^s \int \widehat{\rho_T}(\lambda - \tau) \widetilde{F}(\lambda, \xi) d\lambda \right\|_{L^2_{\tau, \xi}(\Omega_2)} \lesssim T^\epsilon \|F\|_{Z_+^{s, b-1+\epsilon}}$$

and consequently the result follows. \square

We remark that the factor $T^{\frac{1}{2}-b}$ in front of the term $\|\psi\|_{Y_\pm^{s,0}}$ in (16) is not ideal as for T small, this will blow up since $b > \frac{1}{2}$. This cannot be avoided, as a simple scaling argument shows that this is in fact the best possible exponent on T . Essentially the problem comes because the spaces $Y^{s,0}$ and $Z^{s,b}$ scale differently, more precisely the $Y^{s,0}$ space scales like $Z^{s,b}$ at the endpoint $b = \frac{1}{2}$. However, the term $T^{\frac{1}{2}-b}$ is not a huge problem, as if we can take b sufficiently close to $\frac{1}{2}$, then we can safely absorb this into the inhomogeneous term $T^\epsilon \|F\|_{Y_\pm^{s, \epsilon-1}}$ in Lemma 9.

Corollary 11. *Let $-\frac{1}{2} < s < 0$, $0 < \epsilon < \frac{1}{6}$, and $\frac{1}{2} < b < \min\{1 + \epsilon, 1 + s\}$. Assume $0 < T < 1$ and define $S_T = [-T, T] \times \mathbb{R}$. Let ψ be the solution to*

$$\partial_t \psi \pm \partial_x \psi = F$$

with $\psi(0) = f$. Then

$$\|\psi\|_{Z_\pm^{s,b}(S_T)} \lesssim T^{\frac{1}{2}-b} \|f\|_{H^s} + T^\epsilon \inf_{F'=F \text{ on } S_T} \left(\|F'\|_{Y_\pm^{s, -1+2\epsilon}} + \|F'\|_{Z_\pm^{s, b-1+2\epsilon}} \right).$$

Proof. Follows from Lemma 9 and Proposition 10. \square

We now come to the proof of Proposition 8.

Proof of Proposition 8. It is enough to consider the case $\frac{-1}{2} < r \leq s < 0$. Choose $\epsilon > 0$ sufficiently small such that

$$r > \frac{-1}{2} + 4\epsilon \quad (18)$$

and

$$\frac{1}{2} < b < \frac{1}{2} + \epsilon. \quad (19)$$

A standard argument using Corollary 11 reduces the problem to obtaining the estimates

$$\|\psi\phi\|_{Z_{\pm}^{s,b-1+2\epsilon}} \lesssim \|\psi\|_{Z_{\pm}^{s,b}} \|\phi\|_{Z_{\mp}^{r,b}}, \quad (20)$$

$$\|\psi\phi\|_{Y_{\pm}^{s,2\epsilon-1}} \lesssim \|\psi\|_{Z_{\pm}^{s,b}} \|\phi\|_{Z_{\mp}^{r,b}}, \quad (21)$$

$$\|\psi\|_{Y_{\pm}^{s,2\epsilon-1}} \lesssim \|\psi\|_{Z_{\mp}^{s,b}}. \quad (22)$$

We start with (20). By Lemma 5 we need

$$s \prec \{s, b\}, \quad b - 1 + 2\epsilon \prec \{b, r\}.$$

The first condition is straight forward since $s > \frac{-1}{2}$ and $b > \frac{1}{2}$. For the second we need

$$b + r \geq 0, \quad b - 1 + 2\epsilon \leq \min\{b, r\}, \quad b - 1 + 2\epsilon < b + r - \frac{1}{2}$$

which all hold in view of the assumptions (18) and (19).

To prove (21), we observe that by an application of the triangle inequality on the Fourier transform side, it suffices to show that

$$\|\psi\phi\|_{Y_{\pm}^{s,-1}} \lesssim \|\psi\|_{Z_{\pm}^{s,b-2\epsilon}} \|\phi\|_{Z_{\mp}^{r-2\epsilon,b}}.$$

By letting $a_0 = s$ and $b_0 = r - 4\epsilon$ in Lemma 4, we can reduce this to showing

$$\begin{aligned} s \prec \{s, b\}, \quad r - 4\epsilon \prec \{b - 2\epsilon, r - 2\epsilon\}, \quad s \prec \{s, r + 1 - 4\epsilon\} \\ s + b - 2\epsilon > \frac{-1}{2}, \quad r + b - 2\epsilon > \frac{-1}{2}. \end{aligned} \quad (23)$$

The first condition is obvious. For the second condition we need

$$b - 2\epsilon + r - 2\epsilon \geq 0, \quad r - 4\epsilon \leq \min\{b - 2\epsilon, r - 2\epsilon\}, \quad r - 4\epsilon < b - 2\epsilon + r - 2\epsilon - \frac{1}{2}$$

which all follow from (18) and (19). The third condition in (23) can be written as

$$s + r + 1 - 4\epsilon \geq 0, \quad s \leq \min\{s, r + 1 - 4\epsilon\}, \quad s < s + r + 1 - 4\epsilon - \frac{1}{2}$$

and again each of these inequalities follows from (18), (19) and $r \leq s < 0$. The remaining conditions in (23) are also easily seen to be satisfied and so (21) follows.

Finally to prove (22) we use Holder's inequality to obtain

$$\begin{aligned} \|\psi\|_{Y_{\pm}^{s,2\epsilon-1}} &= \|\langle \xi \rangle^s \int_{\mathbb{R}} \langle \tau \pm \xi \rangle^{2\epsilon-1} |\widehat{\psi}| d\tau\|_{L_{\xi}^2} \lesssim \|\langle \xi \rangle^s \widehat{\psi}\|_{L_{\tau,\xi}^2} \\ &\lesssim \|\langle \tau \mp \xi \rangle^s \langle \tau \pm \xi \rangle^{|s|} \widehat{\psi}\|_{L_{\tau,\xi}^2} \\ &\leq \|\psi\|_{Z_{\mp}^{s,b}}. \end{aligned}$$

□

5. GLOBAL EXISTENCE

Here we prove Corollary 2.

Proof of Corollary 2. The persistence of regularity in Theorem 1 shows that it suffices to prove global existence in the case $s = 0$ and $\frac{-1}{2} < r \leq 0$. Let (u_{\pm}, A_{\pm}) be the solution to (5) and (6) given by Theorem 1 with initial data $(f_{\pm}, a_{\pm}) \in L^2 \times H^r$. We extend (u_{\pm}, A_{\pm}) to some maximal interval of existence $(-T, T)$. To show the solution is global in time, it is enough to show that if $T < \infty$ then we have the bound

$$\sup_{t \in (-T, T)} \|A_{\pm}(t)\|_{H^r} < \infty. \quad (24)$$

Since supposing (24) holds, we can extend the solution past $(-T, T)$ by using the L^2 conservation of u_{\pm} , together with the local well-posedness of Theorem 1. Thus contradicting the fact that $(-T, T)$ was the maximal time of existence. Consequently we must have $T = \infty$.

To obtain the bound (24) we make use of the following decomposition first used in [5] based on an idea due to Delgado [7]. We split the Dirac component of our solution u_{\pm} into a mass free part u_{\pm}^L satisfying

$$\begin{aligned} i(\partial_t u_{\pm}^L \pm \partial_x u_{\pm}^L) &= -A_{\mp} u_{\pm}^L \\ u_{\pm}^L(0) &= f_{\pm} \end{aligned}$$

and a term u_{\pm}^N with vanishing initial data

$$\begin{aligned} i(\partial_t u_{\pm}^N \pm \partial_x u_{\pm}^N) &= m u_{\mp} - A_{\mp} u_{\pm}^N \\ u_{\pm}^N(0) &= 0. \end{aligned}$$

Observe that $u_{\pm} = u_{\pm}^L + u_{\pm}^N$. Since A_{\pm} is real valued, a computation shows that

$$\partial_t |u_{\pm}^L|^2 \pm \partial_x |u_{\pm}^L|^2 = 0$$

and

$$\partial_t |u_{\pm}^N|^2 \pm \partial_x |u_{\pm}^N|^2 = 2m \Im(u_{\mp} \overline{u_{\pm}^N}).$$

Hence

$$|u_{\pm}^L(t, x)| = |f_{\pm}(x \mp t)| \quad (25)$$

and, via the Duhamel formula²,

$$\sup_{|t| < T} (\|u_+^N(t)\|_{L_x^\infty} + \|u_-^N(t)\|_{L_x^\infty}) \lesssim_{T, m} \|f_+\|_{L^2} + \|f_-\|_{L^2}. \quad (26)$$

To obtain the bound (24), we note that the equation for A_{\pm} easily leads to

$$\|A_+(t)\|_{H_x^r} + \|A_-(t)\|_{H_x^r} \lesssim \|a_+\|_{H^r} + \|a_-\|_{H^r} + \int_0^t \|u_+ \overline{u_-}\|_{L_x^2} ds \quad (27)$$

and so it suffices to bound $\int_{|s| < T} \|u_+(s) \overline{u_-}(s)\|_{L_x^2} ds$ in terms of the initial data f_{\pm} . If we now use the decomposition $u_{\pm} = u_{\pm}^L + u_{\pm}^N$ we have

$$u_+ \overline{u_-} = u_+ \overline{u_-}^L + u_+ \overline{u_-}^N = u_+^L \overline{u_-}^L + u_+^N \overline{u_-}^L + u_+ \overline{u_-}^N. \quad (28)$$

²For more detail see Proposition 7 in [5].

The terms involving u_{\pm}^N are straightforward by (26), while for the remaining term Holder's inequality followed by a change of variables gives

$$\int_{|s|<T} \|u_+^L(s)\overline{u}_-^L(s)\|_{L_x^2(\mathbb{R})} ds \lesssim_T \|f_+(x-s)\overline{f}_-(x+s)\|_{L_{s,x}^2(\mathbb{R}^2)} \lesssim \|f_+\|_{L^2} \|f_-\|_{L^2}.$$

Therefore the required bound (24) follows. \square

APPENDIX - PROOF OF (3)

Here we will sketch the proof of (3). This result is essentially well-known, but for the readers convenience we will give the outline of the proof.

Proof of estimate 3. We start by noting that the inequality (3) follows immediately from the estimates

$$\|fg\|_{H^s} \lesssim \|f\|_{B_{2,1}^{\frac{1}{2}}} \|g\|_{H^s} \quad (29)$$

and

$$\|\rho_T(t)\|_{B_{2,1}^{\frac{1}{2}}} \lesssim \|\rho\|_{B_{2,1}^{\frac{1}{2}}} \quad (30)$$

where $\|f\|_{B_{2,1}^{\frac{1}{2}}} = \sum_{N \in 2^{\mathbb{N}}} N^{\frac{1}{2}} \|f_N\|_{L^2}$ and

$$f_N = \widehat{P_N f} = \chi_{\{|\xi| \sim N\}} \widehat{f}$$

for $N > 1$ with $\widehat{f}_1 = \chi_{\{|\xi| \lesssim 1\}} \widehat{f}$. We use χ_{Ω} to denote the characteristic function of the set Ω . We also use the notation $\widehat{f}_{\ll N} = \chi_{\{|\xi| \ll N\}} \widehat{f}$. To prove (29) we recall the characterisation

$$\|f\|_{H^s}^2 \approx \sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_N\|_{L^2}^2.$$

as well as the Trichotomy formula

$$P_N(fg) \approx f_{\ll N} g_N + f_N g_{\ll N} + \sum_{M \gtrsim N} P_N(f_M g_M)$$

where the sum is over dyadic numbers $M \in 2^{\mathbb{N}}$. We estimate each of these terms separately. For the first term we observe that

$$\|f_{\ll N} g_N\|_{L^2} \lesssim \|\widehat{f}_{\ll N}\|_{L^1} \|\widehat{g}_N\|_{L^2} \lesssim \left(\sum_{M \ll N} M^{\frac{1}{2}} \|f_M\|_{L^2} \right) \|g_N\|_{L^2}$$

and so

$$\begin{aligned} \sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_{\ll N} g_N\|_{L^2}^2 &\lesssim \sum_{N \in 2^{\mathbb{N}}} N^{2s} \left(\sum_{M \ll N} M^{\frac{1}{2}} \|f_M\|_{L^2} \right)^2 \|g_N\|_{L^2}^2 \\ &\lesssim \left(\sum_{M \in 2^{\mathbb{N}}} M^{\frac{1}{2}} \|f_M\|_{L^2} \right)^2 \sum_{N \in 2^{\mathbb{N}}} N^{2s} \|g_N\|_{L^2}^2 \approx \|f\|_{B_{2,1}^{\frac{1}{2}}}^2 \|g\|_{H^s}^2. \end{aligned}$$

To estimate the term $f_N g_{\ll N}$ a similar computation gives

$$\sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_N g_{\ll N}\|_{L^2}^2 \lesssim \sum_{N \in 2^{\mathbb{N}}} N^{2s} \left(\sum_{M \ll N} M^{\frac{1}{2}} \|g_M\|_{L^2} \right)^2 \|f_N\|_{L^2}^2.$$

Now since $s < \frac{1}{2}$, we have

$$\left(\sum_{M \ll N} M^{\frac{1}{2}} \|g_M\|_{L^2} \right)^2 \lesssim \left(\sum_{M \ll N} M^{1-2s} \right) \left(\sum_{M \ll N} M^{2s} \|g_M\|_{L^2}^2 \right) \lesssim N^{1-2s} \|g\|_{H^s}^2$$

and therefore

$$\sum_{N \in 2^{\mathbb{N}}} N^{2s} \|f_N g_{\ll N}\|_{L^2}^2 \lesssim \|g\|_{H^s}^2 \sum_{N \in 2^{\mathbb{N}}} N \|f_N\|_{L^2}^2 \approx \|f\|_{H^{\frac{1}{2}}}^2 \|g\|_{H^s}^2 \lesssim \|f\|_{B_{2,1}^{\frac{1}{2}}}^2 \|g\|_{H^s}^2.$$

Finally, for the remaining term $\sum_{M>N} P_N(f_M g_M)$, we note that

$$\begin{aligned} \left\| \sum_{M>N} P_N(f_M g_M) \right\|_{L^2} &\lesssim \sum_{M>N} \|P_N(f_M g_M)\|_{L^2} \\ &\lesssim \sum_{M>N} N^{\frac{1}{2}} \|f_M\|_{L^2} \|g_M\|_{L^2} \\ &\lesssim N^{\frac{1}{2}} \left(\sum_{M>N} M^{-2s} \|f_M\|_{L^2}^2 \right)^{\frac{1}{2}} \|g\|_{H^s}. \end{aligned}$$

Hence, for $s > \frac{-1}{2}$,

$$\begin{aligned} \sum_{N \in 2^{\mathbb{N}}} N^{2s} \left\| \sum_{M>N} P_N(f_M g_M) \right\|_{L^2}^2 &\lesssim \|g\|_{H^s}^2 \sum_{N \in 2^{\mathbb{N}}} N^{2s+1} \sum_{M>N} M^{-2s} \|f_M\|_{L^2}^2 \\ &\lesssim \|g\|_{H^s}^2 \sum_{M \in 2^{\mathbb{N}}} M^{-2s} \|f_M\|_{L^2}^2 \sum_{N<M} N^{1+2s} \\ &\lesssim \|g\|_{H^s}^2 \sum_{M \in 2^{\mathbb{N}}} M \|f_M\|_{L^2}^2 \\ &\lesssim \|g\|_{H^s}^2 \|f\|_{B_{2,1}^{\frac{1}{2}}}^2 \end{aligned}$$

and so (29) follows.

The inequality (30) follows by using the characterisation³

$$\|f\|_{B_{2,1}^{\frac{1}{2}}} \approx \|f\|_{L^2} + \int_{\mathbb{R}} \|f(x) - f(x-y)\|_{L_x^2} \frac{dy}{|y|^{1+\frac{1}{2}}}$$

together with a change of variables. □

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³See for instance Theorem 7.47 in [1].

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